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Semisimplicial Unital Groups

David J. Foulis[1] **and Richard J. Greechie**[2]

If there is an order-preserving group isomorphism from a directed abelian group G with a finitely generated positive cone G^+ onto a simplicial group, then G is called a semisimplicial group. By factoring out the torsion subgroup of a unital group having a finite unit interval, one obtains a semisimplicial unital group. We exhibit a representation for the base-normed space associated with a semisimplicial unital group G as the Banach dual space of a finite dimensional order-unit space that contains G as an additive subgroup. In terms of this representation, we formulate necessary and sufficient conditions for G to be archimedean.

KEY WORDS: partially ordered abelian group; directed group; unital group; simplicial group; semisimplicial unital group; effect algebra; state; base-normed space, order-unit space.

1. INTRODUCTION

If $\mathcal{H}$ is a Hilbert space, then the set $\mathbb{G}$ of all bounded self-adjoint operators on $\mathcal{H}$ is organized in the usual way into a partially ordered real linear space (in fact, an order-unit Banach space), and the operators belonging to the interval $\mathbb{E} := \{A \in \mathbb{G} \mid \mathbf{0} \leq A \leq \mathbf{1}\}$ are called the *effect operators* on $\mathcal{H}$. The set $\mathbb{E}$, organized into a partial algebra under the restriction $\oplus$ of addition on $\mathbb{G}$ to $\mathbb{E}$, is called the *algebra of effect operators on* $\mathcal{H}$. In the contemporary theory of quantum measurement (Busch *et al.*, 1991), an observable is represented by a positive-operator-valued (POV) measure defined on a σ-field of sets and taking on values in the effect algebra $\mathbb{E}$.

Abstraction of the salient features of the Hilbert-space effect algebra $\mathbb{E}$ has led to the definition and study of more general *effect algebras* (Foulis and Bennett, 1994), especially effect algebras that, like $\mathbb{E}$, arise as intervals in partially ordered linear spaces (Gudder *et al.*, 1999), or more generally, in partially ordered abelian groups (Bennett and Foulis, 1997). The Hilbert-space effect

[1] Emeritus Professor of Mathematics and Statistics, University of Massachusetts, Amherst, MA 01003; 1 Sutton Court, Amherst, MA 01002, USA; e-mail: foulis@math.umass.edu.
[2] Department of Mathematics and Statistics, College of Engineering and Science, Louisiana Tech University, Ruston, LA 71272, USA.

algebra $\mathbb{E}$ harbors many mysteries, and some aspects of the POV theory of quantum measurement are resistant to satisfactory phenomenological interpretation. The development in (Foulis and Gudder, 2001) suggests that a study of simpler effect algebras, for instance, finite effect algebras, might cast some light on these matters.

Thus, in this paper, we shall be studying intervals in the positive cones of partially ordered abelian groups, paying special attention to the case in which the intervals are finite. Suppose H is a partially ordered abelian group with positive cone $H^+, u \in H^+$, and E is the interval $E = \{e \in H \mid 0 \le e \le u\}$. As our primary concern is with the structure of the interval E, nothing is lost if we drop down to the subcone C of H^+ generated by E and then to the subgroup $G := C - C$ of H. Then G can be organized into a partially ordered abelian group with positive cone $G^+ := C$, and we have $E = \{e \in G \mid 0 \le e \le u\}$. Therefore, the passage from H to G does not affect the structure of E as an effect algebra, but now E generates G^+ as a cone, and G^+ generates G as a group. We refer to such a G as a *unital group* with *unit u* and *unit interval E*.

Suppose G is a unital group with a finite unit interval E. Then G is finitely generated, its torsion subgroup G_τ is a direct summand, and the quotient group G/G_τ is a free abelian group of finite rank. The natural group epimorphism $\eta: G \to G/G_\tau$ can be used to transfer the partial order on G to a partial order on G/G_τ, whence G/G_τ has the structure of a torsion-free unital group with $\eta(E)$ as its (obviously finite) unit interval. It turns out that a torsion-free unital group with a finite unit interval is a *semisimplicial* unital group, i.e., it admits an order-preserving group isomorphism onto a simplicial group. Conversely, every semisimplicial unital group is torsion free and has a finite unit interval. Thus, in what follows, semisimplicial unital groups will be the primary objects of our study.

2. REVIEW OF BASIC CONCEPTS

The abelian groups that we consider will be written using additive notation. The positive cone in a partially ordered abelian group G is denoted by $G^+ := \{g \in G \mid 0 \le g\}$. If $G = G^+ - G^+$, then G is said to be *directed*. If $A \subseteq G^+$ and every element in G^+ is a finite linear combination with nonnegative integer coefficients of elements of A, then we say that A *generates* G^+. The positive cone G^+ is said to be *finitely generated* iff it is generated by a finite set $A \subseteq G^+$. An element $u \in G^+$ is called an *order unit* iff, for each element $g \in G$, there exists a positive integer n such that $g \le nu$. If there exists an order unit u in G, then G is directed. If H is a subgroup of G, then H forms a partially ordered abelian group under the *induced partial order* obtained by restriction of the partial order on G to H, and the corresponding *induced positive cone* is $H^+ = H \cap G^+$.

As an additive group, and totally ordered in the usual way, the system $\mathbb{R}$ of real numbers forms a directed abelian group with the *standard positive cone* $\mathbb{R}^+ = \{x^2 \mid x \in \mathbb{R}\}$. The systems $\mathbb{Q}$ and $\mathbb{Z}$ of rational numbers and integers, respectively, form directed additive subgroups of $\mathbb{R}$ with the induced total orders and with the *standard positive cones* $\mathbb{Q}^+ = \mathbb{Q} \cap \mathbb{R}^+$ and $\mathbb{Z}^+ = \mathbb{Z} \cap \mathbb{R}^+$. In each of the additive abelian groups $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Z}$ with the standard total order, every strictly positive element is an order unit. Only $\mathbb{Z}$ has a finitely generated positive cone.

A partially ordered abelian group G is *lattice ordered* iff, as a partially ordered set, it forms a lattice (i.e., any two elements $a, b \in G$ have an infimum $a \wedge b$ and a supremum $a \vee b$). An *interpolation group* is a partially ordered abelian group G such that, for $a, b, c, d \in G$ with $a, b \leq c, d$ (i.e., $a \leq b, a \leq d, b \leq c$, and $b \leq d$), there exists $t \in G$ with $a, b \leq t \leq c, d$ (Goodearl, 1986, Chapter 2). If G is lattice ordered, it is an interpolation group.

The partially ordered abelian group G is said to be *archimedean* iff, for all $a, b \in G$, the condition that $na \leq b$ for all positive integers n implies that $-a \in G^+$. If, for all $a \in G$, the existence of a positive integer n such that $na \in G^+$ implies that $a \in G^+$, then G is said to be *unperforated*. If G is archimedean and directed, then it is unperforated (Goodearl, 1986, Proposition 1.24). Also, if G is lattice ordered, it is unperforated (Goodearl, 1986, Proposition 1.22), and if G is unperforated, then it is obviously torsion free (i.e., there are no nonzero elements of finite order in G).

If G and H are partially ordered abelian groups, a group homomorphism $\phi\colon G \to H$ is order preserving (i.e., $g_1 \leq g_2 \Rightarrow \phi(g_1) \leq \phi(g_2)$ for all $g_1, g_2 \in G$) iff $\phi(G^+) \subseteq H^+$. If there is a group isomorphism $\phi\colon G \to H$ of G onto H such that both ϕ and ϕ^{-1} are order preserving (i.e., $\phi(G^+) = H^+$), then G and H are *isomorphic as partially ordered abelian groups*.

Under coordinatewise partial order, the cartesian product of partially ordered abelian groups is again a partially ordered abelian group. In particular, if r is a positive real number, the r-fold cartesian product $\mathbb{Z}^r = \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ with coordinatewise partial order is a lattice-ordered abelian group with the *standard positive cone* $(\mathbb{Z}^+)^r = \mathbb{Z}^+ \times \mathbb{Z}^+ \times \cdots \times \mathbb{Z}^+$. If G is isomorphic as a partially ordered abelian group to $\mathbb{Z}^r$ with the standard positive cone $(\mathbb{Z}^+)^r$, then G is called a *simplicial group* (Goodearl, 1986, p. 47). By (Goodearl, 1986, Corollary 3.14), G is simplicial iff it is an interpolation group with an order unit and the positive cone G^+ satisfies the descending chain condition (i.e, there does not exist an infinite strictly descending chain in G^+.)

3. UNITAL GROUPS AND UNIGROUPS

Let G be a partially ordered abelian group and let $u \in G^+$. We define the *interval* $G^+[0, u] := \{g \in G \mid 0 \leq g \leq u\}$ and regard $G^+[0, u]$ as a bounded partially ordered set under the restriction of the partial order on G. The interval $G^+[0, u]$ carries the natural order-reversing involution $e \mapsto u - e$ for $e \in G^+[0, u]$. If G^+ $[0, u]$ generates G^+, then u is said to be *generative*. If G is directed, then every generative element in G^+ is an order unit. However, as the following example shows, order units in G^+ need not be generative.

Example 1. Let $G = \mathbb{Z} \times \mathbb{Z}$ as an abelian group. The *strict* partial order $\ll$ on G is defined by $(a, b) \ll (c, d)$ iff $a = c, b = d$ or $a < c, b < d$. With the strict partial order, G is a directed unperforated abelian group, and every nonzero element in the positive cone G^+ is an order unit. Also, G^+, being a subcone of $\mathbb{Z}^+ \times \mathbb{Z}^+$, satisfies the descending chain condition. However, no element in G^+ is generative, G^+ is not finitely generated, and G is not archimedean.

A *unital group* is a partially ordered abelian group G with a specified generative order unit u, called the *unit* of G (Foulis, 2003, Definition 2.3 (ii)). Evidently, every unital group is directed. If G is a unital group with unit u, then $G^+[0, u]$ is called the *unit interval* in G. If E is the unit interval in the unital group G, then E generates G^+ and $G = G^+ - G^+$, so E is a set of generators for G as an abelian group.

Example 2. Suppose U is a partially ordered linear space over $\mathbb{R}$ and u is an order unit in U (Alfsen, 1971, p. 68). Then, neglecting multiplication of vectors in U by non-integer scalars, we can regard U as a partially ordered abelian group. As such, U is a unital group with unit u. (See Cook and Foulis (2004), Lemma 3.4).

Let G be a unital group with unit u and unit interval $E = G^+[0, u]$. If K is an abelian group, then a mapping $\phi: E \to K$ is called a *K-valued measure* on (or for) E iff, whenever $e, f, e + f \in E$, we have $\phi(e + f) = \phi(e) + \phi(f)$. If $\Phi: G \to K$ is a group homomorphism, then the restriction $\phi := \Phi|_E$ of Φ to E is a K-valued measure on E. Owing to the fact that E is a set of generators for the group G, a group homomorphism $\Phi: G \to K$ is uniquely determined by the K-valued measure $\phi = \Phi|_E$. If every K-valued measure on E can be extended to a (necessarily unique) group homomorphism $\Phi: G \to K$, then G is called a *K-unital* group. If G is a K-unital group for every abelian group K, then G is called a *unigroup* (Foulis *et al.*, 1998). If H is a partially ordered abelian group, then an H-valued measure ϕ on E is called an *H^+-valued measure* iff $\phi(E) \subseteq H^+$. If every H^+-valued measure on E can be extended to a group homomorphism $\Phi: G \to H$, then G is called an *H^+-unital* group. Evidently, G is a unigroup $\Rightarrow$ G is H-unital $\Rightarrow$ G is H^+-unital.

Let G and H be unital groups with units u and v and unit intervals E and L, respectively. An order-preserving group homomorphism $\Psi: G \to H$ such that $\Psi(u) = v$ is called a *unital morphism*. A bijective unital morphism Ψ from G onto H such that $\Psi^{-1}: H \to G$ is also a unital morphism is called a *unital isomorphism*.

A mapping $\psi: E \to L$ is called an *effect-algebra morphism* iff $\psi(u) = v$ and, regarded as a mapping $\psi: E \to H$, it is an H-valued measure. If $\Psi: G \to H$ is a unital morphism, then the restriction $\psi := \Psi|_E$ of Ψ to E is an effect-algebra morphism. A bijective effect-algebra morphism $\psi: E \to L$ such that $\psi^{-1}: L \to E$ is also an effect-algebra morphism is called an *effect-algebra isomorphism* of E onto L.

Suppose G is a unital group with unit interval E. Then there is a unigroup $\mathcal{G}$ with unit interval $\mathcal{E}$ and there is a surjective unital morphism $\xi: \mathcal{G} \to G$ such that the restriction $\xi|_{\mathcal{E}}$ is an effect-algebra isomorphism of $\mathcal{E}$ onto E (Bennett and Foulis, 1997, Corollary 4.2). Thus, unigroups are universal unital groups in the sense that every unital group is the image of a unigroup under a surjective unital morphism that restricts to an effect-algebra isomorphism on the unit interval.

Every unital interpolation group is a unigroup, so every lattice-ordered unital group is a unigroup. Every unital group that is obtained from a partially ordered linear space with an order unit as in Example 2 is a unigroup. In particular, the order-unit normed linear space $\mathbb{G}$ of bounded self-adjoint operators on a Hilbert space $\mathcal{H}$, regarded as an additive unital group, is an archimedean unigroup. However, unless $\mathcal{H}$ is one-dimensional, $\mathbb{G}$ does not form an interpolation group.

If G is a simplicial group, there is a uniquely determined smallest order unit u in G, the order units in G are precisely the elements $v \in G$ with $u \leq v$, and with any such v as unit, G forms a unigroup. More generally, if the unit u in an interpolation unigroup G is a minimal order unit, then G is called a *Boolean* unigroup. If E is the unit interval in a Boolean unigroup with unit u, then E forms a Boolean algebra (a bounded, complemented, distributive lattice) in such a way that, for $e \in E$, $u - e$ is the Boolean complement of e. Conversely, every Boolean algebra B is isomorphic (as a Boolean algebra) to the unit interval in a Boolean unigroup that is uniquely determined up to a unital isomorphism.

The additive group $\mathbb{R}$ of real numbers with the standard (total) order forms a unigroup with 1 as a unit and with the standard unit interval $\mathbb{R}^+[0, 1]$ as its unit interval. If G is a unital group, then a unital morphism $\omega: G \to \mathbb{R}$ is called a *state* on (or for) G (Alfsen, 1971, p. 72), (Goodearl, 1986, Chapter 4). We denote by $\Omega(G)$ the set of all states on G. If the only element $g \in G$ such that $\omega(g) = 0$ for all $\omega \in \Omega(G)$ is $g = 0$, then $\Omega(G)$ *separates the points in* G, i.e., if $g, h \in G$ and $g \neq h$, there exists $\omega \in \Omega(G)$ with $\omega(g) \neq \omega(h)$. If G_τ is the torsion subgroup of G, then $\omega(q) = 0$ for all $\omega \in \Omega(G)$ and all $q \in G_\tau$. Consequently, if $\Omega(G)$ separates the points of G, then G is torsion free.

Let G be a unital group with unit interval E. A *probability measure* on (or for) E is an effect-algebra morphism $\mu: E \to \mathbb{R}^+[0, 1]$. If $\omega \in \Omega(G)$, then the restriction $\omega|_E$ of the state ω to E is a probability measure on E. If G is $\mathbb{R}^+$-unital, then there is a bijective correspondence $\omega \leftrightarrow \mu$ between states $\omega \in \Omega(G)$ and probability measures μ on E given by $\mu = \omega|_E$. The probability measures on the unit interval E of a Boolean unigroup G are just the finitely additive probability measures (in the usual sense) on the Boolean algebra E.

4. THE BASE-NORMED BANACH SPACE $V(G)$

The notions sketched in this section are developed in more detail in (Cook and Foulis (2004)). Let G be a unital group with unit u and unit interval E. The set $\mathbb{R}^G$ of all functions $\nu: G \to \mathbb{R}$ forms a locally convex Hausdorff linear topological space over $\mathbb{R}$ with pointwise operations and with the topology of pointwise convergence. Denote by hom$(G, \mathbb{R})$ the closed linear subspace of $\mathbb{R}^G$ consisting of the additive group homomorphisms $\nu: G \to \mathbb{R}$ and let hom$(G, \mathbb{R})^+$ be the subset of hom$(G, \mathbb{R})$ consisting of the order-preserving group homomorphisms. Then hom$(G, \mathbb{R})$ forms a partially ordered linear space over $\mathbb{R}$ with the partial order relation $\leq^+$ defined for $\nu_1, \nu_2 \in$ hom$(G, \mathbb{R})$ by $\nu_1 \leq^+ \nu_2$ iff $\nu_2 - \nu_1 \in$ hom$(G, \mathbb{R})^+$.

In general, the partially ordered linear space hom$(G, \mathbb{R})$ will not be directed, e.g., hom$(\mathbb{R}, \mathbb{R})$ is not directed because of the existence of discontinuous additive group homomorphisms $\nu: \mathbb{R} \to \mathbb{R}$. But the linear subspace $V(G) :=$ hom$(G, \mathbb{R})^+ -$ hom$(G, \mathbb{R})^+$ is a directed linear space over $\mathbb{R}$ with positive cone $V(G)^+ =$ hom$(G, \mathbb{R})^+$ under the restriction of the partial order $\leq^+$. If G has a finite unit interval E, then $V(G) =$ hom$(G, \mathbb{R})$ (Cook and Foulis (2004), Lemma 5.2 (ii)). If G is a unital interpolation group, then $V(G)$ is a Dedekind complete lattice-ordered linear space (Goodearl, 1986, Corollary 2.28).

The set $\Omega(G)$ of all states $\omega: G \to \mathbb{R}$ is a nonempty convex and compact subset of the linear topological space $\mathbb{R}^G$ (Goodearl, 1986, Corollary 4.4 and Proposition 6.2). Furthermore, $\Omega(G)$ is a cone base for the positive cone $V(G)^+$ and, with $\Omega(G)$ as cone base, $V(G)$ forms a base-normed Banach space (Cook and Foulis (2004), Theorem 4.1). The Banach dual space $V(G)^*$ of $V(G)$ is an order-unit Banach space with order unit $e_1 \in V(G)^{*+}$ uniquely determined by the condition that $e_1(\omega) = 1$ for all $\omega \in \Omega(G)$ (Alfsen, 1971, Theorem II.1.15).

Regarding the order-unit Banach space $V(G)^*$ as a unital group as in Example 2, we find that there is a unital morphism $g \mapsto \widehat{g}$ from G into $V(G)^*$ such that, for all $\nu \in V(G)$ and all $g \in G$, $\widehat{g}(\nu) = \nu(g)$ (Cook and Foulis (2004), Theorem 4.3 (i)). The kernel of $g \mapsto \widehat{g}$ is $\{g \in G \mid \omega(g) = 0, \forall \omega \in \Omega(G)\}$. If G has a finite unit interval, then the kernel of $g \mapsto \widehat{g}$ is the torsion subgroup G_τ of the abelian group G (Cook and Foulis (2004), Theorem 5.3).

We define $\widehat{G} := \{\widehat{g} \mid g \in G\}$ to be the image of G under the unital morphism $g \mapsto \widehat{g}$. Then by (Cook and Foulis (2004), Theorem 4.3 (ii)), $\widehat{G}$ can be organized into a unital group with positive cone $\widehat{G}^+ := \{\widehat{g} \mid g \in G^+\}$, with unit $\widehat{u}$, and with $\widehat{E} := \{\widehat{g} \mid g \in G^+$ and $\widehat{u} - \widehat{g} \in \widehat{G}^+\}$ as its unit interval. There is an isomorphism of base-normed spaces $\nu \mapsto \widehat{\nu}$ from $V(G)$ onto $V(\widehat{G})$ such that, for all $g \in G$, $\widehat{\nu}(\widehat{g}) = \nu(g)$ (Cook and Foulis (2004), Theorem 4.3). Thus, in the passage from G to $\widehat{G}$, the associated base-normed and order-unit Banach spaces remain essentially the same.

If there are enough states in $\Omega(G)$ to separate the points in G, then $g \mapsto \widehat{g}$ is a unital isomorphism of G onto $\widehat{G}$. In the contrary case, the passage from G to $\widehat{G}$ is often advantageous because there are always enough states in $\Omega(\widehat{G})$ to separate the points in $\widehat{G}$, whence $\widehat{G}$ is torsion free. Also, if the unit interval in G is finite, so is the unit interval in $\widehat{G}$. Thus, in what follows, we shall be especially interested in torsion free unital groups with finite unit intervals.

5. SEMISIMPLICIAL UNITAL GROUPS

Definition 1. A *semisimplicial group* is a directed partially ordered abelian group G with a finitely generated positive cone G^+ such that there exists an order-preserving group isomorphism from G onto a simplicial group.

We note that the existence of an order-preserving group isomorphism from a directed partially ordered abelian group G onto a simplicial group does not, in itself, imply that the positive cone G^+ is finitely generated (see Example 1).

Lemma 1. *Suppose $G \neq \{0\}$ is a semisimplicial group. Then:*

(i) *G is a torsion-free abelian group with finite positive rank.*
(ii) *If $u \in G^+$, then the interval $G^+[0, u]$ is finite.*
(iii) *There are generative order units in G^+.*
(iv) *The positive cone G^+ satisfies the descending chain condition.*
(v) *If G is a unital group with unit u, then $\Omega(G)$ separates the points in G.*

Proof: By hypothesis there is a nonzero simplicial group Z and an order-preserving group isomorphism $\theta\colon G \to Z$. (i) Z is torsion free and has finite positive rank, so G inherits these properties. Parts (ii) and (iii) follow from (Foulis, 2003, Theorem 2.1). (iv) Let $g_1, g_2 \in G$ with $g_1 > g_2$. Then $\theta(g_1) \geq \theta(g_2)$, but since θ is injective and $g_1 \neq g_2$, it follows that $\theta(g_1) \neq \theta(g_2)$, i.e., $\theta(g_1) > \theta(g_2)$. Thus, the fact that there are no strictly decreasing infinite chains in Z^+ implies that there are no strictly decreasing infinite chains in G^+. Part (v) follows from (Foulis, 2003, Lemma 3.2). □

Corollary 1. *A semisimplicial group is simplicial iff it is an interpolation group.*

Proof: Combine parts (iii) and (iv) of Lemma 1 with (Goodearl, 1986, Corollary 3.14). □

Example 3. A semisimplicial group need not be unperforated. For instance, let $G = \mathbb{Z}$ as an abelian group, but with the nonstandard positive cone $G^+ := \{0, 2, 3, 4, \ldots\}$ obtained by removing 1 from the standard positive cone $\mathbb{Z}^+$. Then

G is directed and $\{2, 3\}$ is a finite set of generators for G^+, but with $g := 1$ we have $2g \in G^+$, $g \notin G^+$.

Theorem 1. *Let $G \neq \{0\}$ be a unital group with unit interval E. Then the following conditions are mutually equivalent:*

(i) *G is semisimplicial.*
(ii) *G is torsion free and E is finite.*
(iii) *There is a positive integer r and a group isomorphism $\theta\colon G \to \mathbb{Z}^r$ such that $\theta(G^+) \subseteq (\mathbb{Z}^+)^r$.*

Proof: That (i) $\Rightarrow$ (ii) follows from Lemma 1 and that (ii) $\Rightarrow$ (iii) follows from (Foulis, 2003, Lemma 3.3). Assume (iii). Since $\mathbb{Z}^r$ is a simplicial group with positive cone $(\mathbb{Z}^+)^r$, we only have to prove that G^+ is finitely generated. If u is the unit in G, then the interval $L := (\mathbb{Z}^+)^r[\mathbf{0}, \theta(u)]$ is finite, $\theta(E) \subseteq L$, and θ is injective, whence E is finite. But E generates G^+, so G^+ is finitely generated. □

In Theorem 1 (iii), the positive integer r is uniquely determined; it is the rank of the abelian group G. However, the group isomorphism θ is determined only up to composition with an order-preserving group automorphism of the simplicial group $\mathbb{Z}^r$. The question of how one chooses an "optimal" θ is of considerable interest, but we shall not pursue it here.

In our subsequent study of a semisimplicial unital group G, it will be convenient to choose θ as in part (iii) of Theorem 1 and use it to *identify G with the abelian group $\mathbb{Z}^r$ in such a way that $G^+ \subseteq (\mathbb{Z}^+)^r$*. This provides a definite computational advantage in that elements of G are vectors $\mathbf{g} = (g_1, g_2, \ldots, g_r)$ with integer components, elements of G^+ have nonnegative integer components, and addition is performed coordinate-wise. Thus, in the sequel, we shall be working with the following data.

Standing Assumptions *For the remainder of this article, we assume that r is a positive integer, $G = \mathbb{Z}^r$ as an additive abelian group, and G is a unital group such that $G^+ \subseteq (\mathbb{Z}^+)^r$.* Denote the unit in G by $\mathbf{u} = (u_1, u_2, \ldots, u_r) \in (\mathbb{Z}^+)^r$ and the unit interval in G by E. We shall use the notation $\leq_G$ for the partial order on the unital group G and the notation $\leq$ for the (point-wise) partial order on the simplicial group $\mathbb{Z}^r$.

Even though $G = \mathbb{Z}^r$ as an abelian group, we have to be careful to distinguish between the partial orders $\leq_G$ and $\leq$. Because $G^+ \subseteq (\mathbb{Z}^+)^r$, we have $\mathbf{g} \leq_G \mathbf{h} \Rightarrow \mathbf{g} \leq \mathbf{h}$ for all $\mathbf{g}, \mathbf{h} \in G = \mathbb{Z}^r$, but the converse will hold for all $\mathbf{g}, \mathbf{h}$ iff G is simplicial.

Suppose $\mathbf{z} \in \mathbb{Z}^r$. Since $\mathbf{u}$ is an order unit in G, there exists a positive integer n such that $\mathbf{z} \leq_G n\mathbf{u}$. But then $\mathbf{z} \leq n\mathbf{u}$, and it follows that $\mathbf{u}$ is an order unit in the simplicial group $\mathbb{Z}^r$. Consequently, the components $u_1, u_2, \ldots, u_r$ of $\mathbf{u}$ are strictly positive integers.

By Theorem 1, the unit interval E in G is finite. By an *atom* in E, we mean an element $\mathbf{0} \neq \mathbf{a} \in E$ such that, for all $\mathbf{b} \in E$, $\mathbf{0} <_G \mathbf{b} \leq_G \mathbf{a} \Rightarrow \mathbf{b} = \mathbf{a}$. Because E is finite, it is *atomic*, i.e., if $\mathbf{0} \neq \mathbf{e} \in E$, there exists at least one atom $\mathbf{a} \in E$ with $\mathbf{a} \leq_G \mathbf{e}$.

Definition 2. Denote by $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ the distinct atoms in E. Let $A_0 = [a_{ij}]$ be the $n \times r$ matrix with the vectors $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ as its successive rows, and let A be the $(n+1) \times r$ matrix obtained from A_0 by appending the vector $\mathbf{u}$ as a final row. A_0 is called the *atom matrix* and A is called the *representation matrix* for E. Let $\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_r}$ be the successive column vectors of A, define C_A to be the column space of A over $\mathbb{R}$, and let C_A^+ be the set of column vectors $\mathbf{c} \in C_A$ such that all components of $\mathbf{c}$ are nonnegative real numbers.

The structure of the semisimplicial unital group G is encoded in the representation matrix A. Indeed, the last row of A is the unit $\mathbf{u}$ and the cone G^+ is the set of all linear combinations with coefficients in $\mathbb{Z}^+$ of the remaining rows. Obviously, the column space C_A of A forms a directed archimedean linear space with positive cone C_A^+. We are going to show that C_A can be organized into a base-normed Banach space that is isomorphic as a base-normed space to $V(G)$.

Definition 3. Let $\mathbf{e_1} = (1, 0, \ldots, 0)$, $\mathbf{e_2} = (0, 1, \ldots, 0), \ldots, \mathbf{e_r} = (0, 0, \ldots, 1)$ be the standard (Kronecker) free basis vectors for the free abelian group $\mathbb{Z}^r$. For $i = 1, 2, \ldots, r$, define $\pi_i \colon G \to \mathbb{Z}$ by $\pi_i(\mathbf{z}) := z_i$ for $\mathbf{z} = (z_1, z_2, \ldots, z_r) \in G = \mathbb{Z}^r$. Define the linear transformation $\alpha \colon V(G) \to C_A$ by $\alpha(\nu) := \sum_{j=1}^{r} \nu(\mathbf{e_j})\mathbf{c_j}$ for $\nu \in V(G)$. Define the linear functional $\epsilon \colon C_A \to \mathbb{R}$ by $\epsilon(\mathbf{c}) := (n+1)$st component of the column vector $\mathbf{c} \in C_A$.

We recall that, since E is finite, $V(G) = \hom(G, \mathbb{R})$. Evidently $\pi_i \in \hom(G, \mathbb{R}) = V(G)$ for $i = 1, 2, \ldots, r$, $\{\pi_1, \pi_2, \ldots, \pi_r\}$ is a vector-space basis for $V(G)$, and $\pi_i(\mathbf{e_j}) = \delta_{ij}$ (Kronecker delta) for $i, j = 1, 2, \ldots, r$.

Theorem 2.

(i) $\{\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}\}$ *is a finite set of generators for the cone* G^+ *and for the abelian group* G.

(ii) *The column vectors* $\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_r}$ *of* A *form a vector-space basis for the column space* C_A, *hence* $\dim(C_A) = r$.

(iii) $\alpha \colon V(G) \to C_A$ *is a linear isomorphism of* $V(G)$ *onto* C_A *such that, for* $\nu \in V(G)$, *the* i*th component of the column vector* $\alpha(\nu)$ *is* $\nu(\mathbf{a_i})$ *for* $i = 1, 2, \ldots, n$, *and* $\epsilon(\alpha(\nu)) = \nu(\mathbf{u})$. *Also,* $\alpha^{-1}(\mathbf{c_i}) = \pi_i$ *for* $i = 1, 2, \ldots, r$.

(iv) C_A *is an archimedean base-normed Banach space over* $\mathbb{R}$ *with cone base* $C_A^+ \cap \epsilon^{-1}(1)$ *and* $\alpha \colon V(G) \to C_A$ *is an isomorphism of base-normed spaces from* $V(G)$ *onto* C_A.

Proof:

Part (i) follows directly from (Foulis, 2003, Lemma 5.1).

(ii) By (i), each vector $\mathbf{e_i} \in G$ is a linear combination with integer coefficients of $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$, and it follows that each vector in $\mathbb{R}^r$ is a linear combination over $\mathbb{R}$ of the rows of the matrix A. Thus, the row space over the field $\mathbb{R}$ of A is $\mathbb{R}^r$, so rank$(A) = r$, and it follows that the r column vectors $\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_r}$ are linearly independent over $\mathbb{R}$. By definition, the column vectors of A span C_A, so they form a vector-space basis for C_A.

(iii) By Definition 3, the ith component of the column vector $\alpha(\nu)$ is $\sum_{j=1}^{r} \nu(\mathbf{e_j})a_{ij} = \nu(\sum_{j=1}^{r} a_{ij}\mathbf{e_j}) = \nu(\mathbf{a_i})$. By the same calculation, the $(n+1)$st component of $\alpha(\nu)$ is $\epsilon(\alpha(\nu)) = \nu(\mathbf{u})$. We have $\alpha(\pi_i) = \sum_{j=1}^{r} \pi_i(\mathbf{e_j})\mathbf{c_j} = \sum_{j=1}^{r} \delta_{ij}\mathbf{c_j} = \mathbf{c_i}$ for $i = 1, 2, \ldots, r$. Since $\pi_i, i = 1, 2, \ldots, r$, is a vector-space basis for $V(G)$, it follows that α is a linear isomorphism of $V(G)$ onto C_A.

(iv) Let $\nu \in V(G)$. Then $\nu \in V(G)^+$ iff $0 \leq \nu(\mathbf{a_i})$ for $i = 1, 2, \ldots, n$. Therefore, since $\mathbf{u} \in G^+$, (iii) implies that $\nu \in V(G)^+$ iff $\alpha(\nu) \in C_A^+$. Also, $\nu \in \Omega(G)$ iff $\nu \in V(G)^+$ with $\nu(\mathbf{u}) = 1$ iff $\alpha(\nu) \in C_A^+$ with $\epsilon(\alpha(\nu)) = 1$. Thus, for the linear isomorphism $\alpha\colon V(G) \to C_A$, we have $\alpha(V(G)^+) = C_A^+$ and $\alpha(\Omega(G)) = C_A^+ \cap \epsilon^{-1}(1)$, whence C_A can be organized into a base-normed space with cone base $C_A^+ \cap \epsilon^{-1}(1)$, and α becomes an isomorphism of base-normed spaces.

By Theorem 2 (i), there is at least one vector $\mathbf{t} = (t_1, t_2, \ldots, t_n) \in (\mathbb{Z}^+)^n$ such that $\mathbf{u} = \sum_{i=1}^{n} t_i \mathbf{a_i}$. Such a vector is called a *multiplicity vector* for the effect algebra E, and there are only finitely many such vectors (Foulis, 2003, Definition 5.1 and ff.). The structure of the effect algebra E is encoded in the set of multiplicity vectors $\mathbf{t}$ for E (Foulis, 2003, Theorem 2.7). □

Definition 4. Let $\mathbf{t_1}, \mathbf{t_2}, \ldots, \mathbf{t_m}$ be the distinct multiplicity vectors for E and let $T = [t_{ij}]$ be the $m \times n$ matrix having these vectors as its successive rows. Define M to be the $m \times (n+1)$ matrix obtained by appending a final column to T consisting entirely of -1's. We call T the *multiplicity matrix* and M the *relation matrix* for E. Define K_M to be the real vector space consisting of all $(n+1)$-dimensional column vectors $\mathbf{c}$ such that $M\mathbf{c} = \mathbf{0}$.

Lemma 2.

(i) *C_A is a linear subspace of K_M.*

(ii) $r = \dim(C_A) \leq \dim(K_M) = n + 1 - rank(M)$.

(iii) $rank(M) = rank(T)$.

Proof:

(i) Since $\sum_{i=1}^{n} t_{ij}\mathbf{a_j} - \mathbf{u} = \mathbf{0}$ for $i = 1, 2, \ldots, m$, it follows that $\mathbf{c_i} \in K_M$ for $i = 1, 2, \ldots, r$, whence $C_A \subseteq K_M$.

(ii) By Theorem 2 (ii), $r = \dim(C_A)$, and by (i), $\dim(C_A) \leq \dim(K_M)$. The vector space K_M can be regarded as the null space of the linear transformation $\mathbf{c} \mapsto M\mathbf{c}$, whence by the rank-plus-nullity theorem $\dim(K_M) = n + 1 - \text{rank}(M)$.

(iii) Part (iii) follows from (Foulis, 2003, Theorem 2.6 (i)). □

Theorem 3. *The following conditions are mutually equivalent:*

(i) *G is $\mathbb{R}$-unital.*
(ii) *G is $\mathbb{Z}$-unital.*
(iii) *G is F-unital for every free abelian group F.*
(iv) *G is G-unital.*
(v) *$n + 1 \leq rank(T) + r$.*
(vi) *$C_A = K_M$.*

Proof: The equivalence of (i), (ii), and (v) follows from (Foulis, 2003, Theorem 5.2) and the fact that $\text{rank}(M) = \text{rank}(T)$. If G is $\mathbb{Z}$-unital and D is a direct product of copies of $\mathbb{Z}$, it is clear that G is D-unital. Also, if G is K-unital and K_1 is a subgroup of K, then G is K_1-unital. As every free abelian group is a subgroup of a direct product of copies of $\mathbb{Z}$, it follows that (ii) ⇒ (iii). As G itself is a free abelian group, we have (iii) ⇒ (iv). As $\mathbb{Z}$ is isomorphic to a subgroup of G, it follows that (iv) ⇒ (ii), whence (i)–(v) are mutually equivalent. The equivalence of (v) and (vi) follows from Lemma 2 (ii), (iii). □

Example 4. Here is an example in which all of the equivalent conditions in Theorem 3 fail. Let G be the simplicial group in Example 3 and organize G into a unital group with unit $u := 5$. Then the unit interval is $E = \{0, 2, 3, 5\}$, the atoms in E are $a_1 := 2, a_2 = 3$, and there is only one multiplicity vector, namely $\mathbf{t} := (1, 1)$. Here $n = 2, r = 1, \text{rank}(T) = 1$, and the condition $n + 1 \leq \text{rank}(T) + r$ in Theorem 3 fails.

In Example 4, G is not archimedean. The authors do not know an example of an archimedean semisimplicial unital group that fails to be $\mathbb{Z}$-unital.

6. THE LINEAR SPACES V AND U

We maintain the assumptions and notation of Section 5. Our purpose in this section is to obtain yet another representation for the base-normed space $V(G)$ as the Banach dual space V of an r-dimensional order-unit space U that contains G

as an additive subgroup. Using this representation, we formulate new necessary and sufficient conditions for G to be archimedean (Theorem 6 and Corollary 3 below).

The group $G = \mathbb{Z}^r$ is an additive subgroup of the r-dimensional coordinate vector space $\mathbb{R}^r$ over $\mathbb{R}$. In what follows, we understand that $\mathbb{R}^r$ is organized into a directed linear space with the *standard positive cone* $(\mathbb{R}^+)^r$ and *coordinate-wise partial order* $\leq$, and also that it is organized into a euclidean space with the usual dot product $\mathbf{a} \cdot \mathbf{b}$ as the inner product.

We have $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}, \mathbf{u} \in E \subseteq G^+ \subseteq (\mathbb{Z}^+)^r \subseteq (\mathbb{R}^+)^r$. The free basis $\mathbf{e_1}$, $\mathbf{e_2}, \ldots, \mathbf{e_r}$ for the abelian group G is an orthonormal basis for the euclidean space $\mathbb{R}^r$, and by Theorem 2 (i), each $\mathbf{e_i}$, $i = 1, 2, \ldots, r$, is a linear combination with integer coefficients of the vectors $\mathbf{a_j}$. Therefore, the atoms $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$ in E span the euclidean space $\mathbb{R}^r$.

We shall be organizing $\mathbb{R}^r$ into a directed linear space in three ways with three different partial orders, one of which is the standard coordinate-wise partial order. In this regard, if P is a real linear space, then we call a subset $C \subseteq P$ a *linear cone* in P iff $0 \in C$, C is closed under addition, C is closed under multiplication by nonnegative real numbers, and $p, -p \in C \Rightarrow p = 0$. If C is a linear cone in P, then there is one and only one way to organize P into a partially ordered linear space with C as its positive cone—namely by defining the partial order $p \leq_C q$ iff $q - p \in C$.

Definition 5.

(i) $V^+ := \{\mathbf{v} \in \mathbb{R}^r \mid 0 \leq \mathbf{v} \cdot \mathbf{a_i} \text{ for } i = 1, 2, \ldots, n\}$.
(ii) $V_\Omega := \{\mathbf{w} \in V^+ \mid \mathbf{w} \cdot \mathbf{u} = 1\}$.
(iii) $U^+ := \left\{ \sum_{j=1}^n y_j \mathbf{a_j} \mid y_j \in \mathbb{R}^+ \text{ for } j = 1, 2, \ldots, n \right\}$.

Lemma 3.

(i) $\mathbf{u} \in E \subseteq G^+ \subseteq U^+ \subseteq (\mathbb{R}^+)^r \subseteq V^+$.
(ii) *Both* V^+ *and* U^+ *are linear cones in* $\mathbb{R}^r$.
(iii) $\mathbf{y} \in \mathbb{R}^r \Rightarrow m\mathbf{u} - \mathbf{y} \in U^+$ *for some positive integer* m.
(iv) $\mathbb{R}^r = V^+ - V^+ = U^+ - U^+$.
(v) *If* $\mathbf{v} \in V^+$ *and* $\mathbf{e} \in E$, *then* $0 \leq \mathbf{v} \cdot \mathbf{e} \leq \mathbf{v} \cdot \mathbf{u}$.
(vi) V_Ω *is a polytope in* $\mathbb{R}^r$ *and it is a cone base for* V^+.
(vii) $U^+ = \{\mathbf{y} \in \mathbb{R}^r \mid 0 \leq \mathbf{y} \cdot \mathbf{v}, \forall \mathbf{v} \in V^+\}$.
(viii) $V^+ = \{\mathbf{v} \in \mathbb{R}^r \mid 0 \leq \mathbf{v} \cdot \mathbf{y}, \forall \mathbf{y} \in U^+\}$.

Proof:

(i) Part (i) is obvious.
(ii) Clearly, both V^+ and U^+ are closed under addition and under multiplication by nonnegative scalars, and $\mathbf{0}$ belongs to both. If $\mathbf{v} \in -V^+ \cap V^+$,

then $\mathbf{v} \cdot \mathbf{a_i} = 0$ for $i = 1, 2, \ldots, n$, so, since the vectors $\mathbf{a_i}$ span $\mathbb{R}^r$, we have $\mathbf{v} = \mathbf{0}$. Thus V^+ is a linear cone in $\mathbb{R}^r$. Because $U^+ \subseteq (\mathbb{R}^+)^r$, we have $-U^+ \cap U^+ \subseteq \{\mathbf{0}\}$, so U^+ is also a linear cone in $\mathbb{R}^r$.

(iii) Let $\mathbf{y} \in \mathbb{R}^r$. Then there are real numbers $y_j \in \mathbb{R}$ such that $\mathbf{y} = \sum_{j=1}^{n} y_j \mathbf{a_j}$. Choose a positive integer k such that $y_j < k$ for $j = 1, 2, \ldots, n$. As $\mathbf{u} - \mathbf{a_j} \in G^+ \subseteq U^+$, we have $k\mathbf{u} - k\mathbf{a_j} = k(\mathbf{u} - \mathbf{a_j}) \in U^+$ for $j = 1, 2, \ldots, n$. Also, as $\mathbf{a_j} \in G^+ \subseteq U^+$ and $k - y_j \in \mathbb{R}^+$, we have $k\mathbf{a_j} - y_j\mathbf{a_j} = (k - y_j)\mathbf{a_j} \in U^+$, whence $k\mathbf{u} - y_j\mathbf{a_j} \in U^+$ for $j = 1, 2, \ldots, n$. Consequently, with $m := nk$ we have $m\mathbf{u} - \mathbf{y} = \sum_{j=1}^{n}(k\mathbf{u} - y_j\mathbf{a_j}) \in U^+$.

(iv) Since $(\mathbb{R}^+)^r \subseteq V^+$ and $\mathbb{R}^r = (\mathbb{R}^+)^r - (\mathbb{R}^+)^r$, it follows that $\mathbb{R}^r = V^+ - V^+$. If $\mathbf{y} \in \mathbb{R}^r$, then by (iii), there is a positive integer m such that $\mathbf{y} = m\mathbf{u} - (m\mathbf{u} - \mathbf{y}) \in U^+ - U^+$, so $\mathbb{R}^r = U^+ - U^+$.

(v) Let $\mathbf{v} \in V^+$ and $\mathbf{e} \in E$. Then $\mathbf{e}$ is a linear combination of $\mathbf{a_i}$, $i = 1, 2, \ldots, n$, with nonnegative integer coefficients, and it follows that $0 \leq \mathbf{v} \cdot \mathbf{a}$. But we also have $\mathbf{u} - \mathbf{e} \in E$, so $0 \leq \mathbf{v} \cdot (\mathbf{u} - \mathbf{e})$, whence $0 \leq \mathbf{v} \cdot \mathbf{e} \leq \mathbf{v} \cdot \mathbf{u}$.

(vi) As V_Ω is an intersection of finitely many closed halfspaces $\{\mathbf{w} \mid 0 \leq \mathbf{w} \cdot \mathbf{a_i}\}$, $i = 1, 2, \ldots, n$, $\{\mathbf{w} \mid 1 \leq \mathbf{w} \cdot \mathbf{u}\}$, and $\{\mathbf{w} \mid 1 \geq \mathbf{w} \cdot \mathbf{u}\}$ of $\mathbb{R}^r$, it is a polyhedron. If $\mathbf{w} \in V_\Omega$, then $0 \leq \mathbf{w} \cdot \mathbf{a_i} \leq \mathbf{w} \cdot \mathbf{u} = 1$ for all $i = 1, 2, \ldots, n$, so the fact that the vectors $\mathbf{a_i}$ span $\mathbb{R}^r$ implies that V_Ω is a bounded subset of the Euclidean space $\mathbb{R}^r$. Therefore, V_Ω is a bounded polyhedron, i.e., a polytope in $\mathbb{R}^r$. Suppose $\mathbf{v} \in V^+$. Then if $\mathbf{v} \cdot \mathbf{u} = 0$, we have $0 \leq \mathbf{v} \cdot \mathbf{a_i} \leq \mathbf{v} \cdot \mathbf{u} = 0$ for $i = 1, 2, \ldots, n$, and, since the vectors $\mathbf{a_i}$ span $\mathbb{R}^r$, it follows that $\mathbf{v} = \mathbf{0}$. Therefore, if $\mathbf{v} \neq \mathbf{0}$, then $t := \mathbf{v} \cdot \mathbf{u} \neq 0$, $\mathbf{w} := (1/t)\mathbf{v} \in V_\Omega$ and $\mathbf{v} = t\mathbf{w}$. Conversely, if t' is a positive real number, $\mathbf{w}' \in V_\Omega$, and $\mathbf{v} = t'\mathbf{w}'$, then $t = \mathbf{v} \cdot \mathbf{u} = t'\mathbf{w}' \cdot \mathbf{u} = t'$ and $\mathbf{w} = (1/t)\mathbf{v} = \mathbf{w}'$. Therefore, V_Ω is a cone base for V^+.

(vii) Part (vii) is a direct consequence of Farkas's lemma (Dax, 1997).

(viii) Part (viii) is obvious. □

Definition 6. Let $V := \mathbb{R}^r$ and $U := \mathbb{R}^r$ as real linear spaces, but organized into directed linear spaces with positive cones V^+ and U^+, respectively. Denote the dual spaces of V and U by V^* and U^*, respectively. Organize V^* and U^* into partially ordered linear spaces with the positive cones $V^{*+} := \{f \in V^* \mid 0 \leq f(\mathbf{v}), \forall \mathbf{v} \in V^+\}$ and $U^{*+} := \{\phi \in U^* \mid 0 \leq \phi(\mathbf{y}), \forall \mathbf{y} \in U^+\}$, respectively. If $\mathbf{v} \in V$, define $\mathbf{v}^\sharp \in U^*$ by $\mathbf{v}^\sharp(\mathbf{y}) := \mathbf{v} \cdot \mathbf{y}$ for all $\mathbf{y} \in U$. If $\mathbf{y} \in U$, define $\mathbf{y}^\natural \in V^*$ by $\mathbf{y}^\natural(\mathbf{v}) := \mathbf{y} \cdot \mathbf{v}$ for all $\mathbf{v} \in V$.

As well as being directed linear spaces, both $V = \mathbb{R}^r$ and $U = \mathbb{R}^r$ are r-dimensional euclidean spaces, and it follows that $\mathbf{v} \mapsto \mathbf{v}^\sharp$ is a linear isomorphism of V onto U^* and $\mathbf{y} \mapsto \mathbf{y}^\natural$ is a linear isomorphism of U onto V^*.

Theorem 4. *V is a base-normed Banach space with V_Ω as cone base and U is an order-unit Banach space with* $\mathbf{u}$ *as order unit. Therefore, V^* is an order-unit Banach space with order unit $e_1 \in V^{*+}$ uniquely determined by the condition $e_1(\mathbf{w}) = 1$ for all $\mathbf{w} \in V_\Omega$, and U^* is a base-normed Banach space with cone base $\Gamma := \{\phi \in U^{*+} \mid \phi(\mathbf{u}) = 1\}$. Under the mapping $\mathbf{v} \mapsto \mathbf{v}^\sharp$, V is isomorphic as a base-normed Banach space to U^*, and under the mapping $\mathbf{y} \mapsto \mathbf{y}^\natural$, U is isomorphic as an order-unit Banach space to V^*.*

Proof: By Lemma 3 (vi), the cone base V_Ω is compact in the Euclidean space $\mathbb{R}^r$, so V is a base-normed space (Alfsen, 1971, Proposition II.1.12). Since V is finite dimensional, the base-norm topology coincides with the Euclidean topology and V is a Banach space under the base norm. By Ellis's theorem (Alfsen, 1971, Theorem II.1.15), the Banach dual space V^* of V is an order unit space with positive cone V^{*+} and with order unit e_1. By Lemma 3 (vii), $U^+ = \{\mathbf{y} \in U \mid \mathbf{y}^\natural(\mathbf{v}) \geq 0, \forall \mathbf{v} \in V^+\}$, whence $\{\mathbf{y}^\natural \mid \mathbf{y} \in U^+\} = V^{*+}$. By Lemma 3 (i) and (iii), $\mathbf{u} \in U^+$ and $\mathbf{u}$ is an order unit in U. Also, $\mathbf{u}^\natural(\mathbf{w}) = \mathbf{u} \cdot \mathbf{w} = 1$ for all $\mathbf{w} \in V_\Omega$, whence $\mathbf{u}^\natural = e_1$. Consequently, U is an order-unit Banach space with order unit $\mathbf{u}$ and as such it is isomorphic to V^* under $\mathbf{y} \mapsto \mathbf{y}^\natural$.

By Ellis's theorem again, the Banach dual space U^* of U is a base-normed space with positive cone U^{*+} and with cone base Γ. By Lemma 3 (viii), $V^+ = \{\mathbf{v} \in V \mid \mathbf{v}^\sharp(\mathbf{y}) \geq 0, \forall \mathbf{y} \in U^+\}$, whence $\{\mathbf{v}^\sharp \mid \mathbf{v} \in V^+\} = U^{*+}$. Also, it is clear that $\mathbf{v} \in V_\Omega \Leftrightarrow \mathbf{v}^\sharp \in U^{*+}$ with $\mathbf{v}^\sharp(\mathbf{u}) = 1 \Leftrightarrow \mathbf{v}^\sharp \in \Gamma$, so V is isomorphic as a base-normed Banach space to U^* under $\mathbf{v} \mapsto \mathbf{v}^\sharp$. □

The unital group G is an additive subgroup of the order-unit Banach space U in such a way that the unit $\mathbf{u}$ of G is the order unit of U, $G^+ \subseteq U^+$, and U is the linear span of E. By the following theorem, this provides an embedding of G into an archimedean unigroup U in such a way that $V(G)$ is isomorphic as a base-normed Banach space to $V(U)$.

Theorem 5. *As a partially ordered abelian group under addition, U is an archimedean unigroup with unit* $\mathbf{u}$ *and, regarded as such, $V(U) = U^*$. If $\nu \in V(U) = U^*$, denote by $\nu|_G$ the restriction of ν to $G \subseteq U$. Then $\nu \mapsto \nu|_G$ is a base-normed-space isomorphism from $V(U)$ onto $V(G)$.*

Proof: We regard U simply as an additive abelian group as in Example 2. By (Cook and Foulis (2004), Lemma 3.4), U is a unigroup with unit $\mathbf{u}$ and $\mathrm{hom}(U, \mathbb{R})^+$ is the set of all $\nu \in U^*$ such that $\nu(U^+) \subseteq \mathbb{R}^+$. Therefore, $V(U)^+ = U^{*+}$, so $V(U) = V(U)^+ - V(U)^+ = U^{*+} - U^{*+}$. By Theorem 4, U^* is directed, hence $V(U) = U^*$. Since U is archimedean as a directed linear space, it is clear that it is archimedean as a unital group.

If $\nu \in V(U) = U^*$, then we have $\nu|_G \in \hom(G, \mathbb{R})$, and since E is finite, $\hom(G, \mathbb{R}) = V(G)$ (Cook and Foulis (2004), Lemma 5.2 (ii)). Obviously, $\nu \mapsto \nu|_G$ is a linear transformation from $V(U)$ into $V(G)$ and, since U is the linear span of G, $\nu \mapsto \nu|_G$ is injective. As G is a free abelian group of rank r, the dimension of $\hom(G, \mathbb{R}) = V(G)$ is r. Also, the dimension of $V(U) = U^*$ is r, so $\nu \mapsto \nu|_G$ is a linear isomorphism of $V(U)$ onto $V(G)$. That this isomorphism maps $V(U)^+ = U^{*+}$ onto $V(G)^+$ and the cone base Γ in U^{*+} onto $\Omega(G)$ is clear. □

Corollary 2. $G^+ \subseteq G \cap U^+ = \{\mathbf{p} \in G \mid 0 \leq \omega(\mathbf{p}), \forall \omega \in \Omega(G)\}$.

Proof: By Theorem 5, if $\mathbf{p} \in G$, then $0 \leq \omega(\mathbf{p})$ for every $\omega \in \Omega(G)$ iff $0 \leq \sigma(\mathbf{p})$ for every $\sigma \in \Omega(U)$. But U is archimedean, so $0 \leq \omega(\mathbf{p})$ for every $\omega \in \Omega(G)$ iff $\mathbf{p} \in U^+$. □

Theorem 6. *The following conditions are mutually equivalent:*

(i) *G is archimedean.*
(ii) *G is unperforated.*
(iii) *For every vector $(q_1, q_2, \ldots, q_n) \in (\mathbb{Q}^+)^n$ such that $\sum_{i=1}^n q_i \mathbf{a_i} \in \mathbb{Z}^r$, there exists a vector $(p_1, p_2, \ldots, p_n) \in (\mathbb{Z}^+)^n$ such that $\sum_{i=1}^n q_i \mathbf{a_i} = \sum_{i=1}^n p_i \mathbf{a_i}$.*
(iv) *$G^+ = G \cap U^+$.*
(v) *There exist $\mathbf{b_j} \in \mathbb{Z}^r$ for $j = 1, 2, \ldots, s$ such that, for all $\mathbf{p} \in \mathbb{Z}^r$, $\mathbf{p} \in G^+ \Leftrightarrow \mathbf{b_j} \cdot \mathbf{p} \geq 0$ for $j = 1, 2, \ldots, s$.*

Proof:

(i) $\Rightarrow$ (ii) by (Goodearl, 1986, Proposition 1.24).
(ii) $\Rightarrow$ (iii). Assume that G is unperforated, and suppose $q_1, q_2, \ldots, q_n \in \mathbb{Q}^+$ such that $\mathbf{p} := \sum_{i=1}^n q_i \mathbf{a_i} \in \mathbb{Z}^r = G$. There is a positive integer M such that $k_i := M q_i \in \mathbb{Z}^+$ for $i = 1, 2, \ldots, n$, and we have $M\mathbf{p} = \sum_{i=1}^n k_i \mathbf{a_i} \in G^+$. Since G is unperforated, it follows that $\mathbf{p} \in G^+$, hence there exist $p_1, p_2, \ldots, p_n \in \mathbb{Z}^+$ such that $\sum_{i=1}^n q_i \mathbf{a_i} = \mathbf{p} = \sum_{i=1}^n p_i \mathbf{a_i}$.
(iii) $\Rightarrow$ (iv). Assume (iii) and let $\mathbf{p} \in G \cap U^+$. Since $\mathbf{p} \in U^+$, there exist $y_i \in \mathbb{R}^+$ for $i = 1, 2, \ldots, n$ such that $\mathbf{p} = \sum_{i=1}^n y_i \mathbf{a_i}$. Since the components of $\mathbf{p}$ as well as those of $\mathbf{a_i}$, $i = 1, 2, \ldots, n$, are integers, it follows that there exist $q_i \in \mathbb{Q}^+$ for $i = 1, 2, \ldots, n$ such that $\mathbf{p} = \sum_{i=1}^n q_i \mathbf{a_i}$. Hence, by (iii), there exist $p_i \in \mathbb{Z}^+$ for $i = 1, 2, \ldots, n$ such that $\mathbf{p} = \sum_{i=1}^n p_i \mathbf{a_i} \in G^+$. Thus, $G \cap U^+ \subseteq G^+$, and the opposite inclusion is obvious.
(iv) $\Rightarrow$ (i). If (iv) holds, then by Corollary 2, $G^+ = \{\mathbf{g} \in G \mid \omega(\mathbf{g}) \geq 0, \forall \omega \in \Omega(G)\}$, whence G is archimedean by (Goodearl, 1986, Theorem 4.14).

(i) $\Leftrightarrow$ (v). The equivalence of (i) and (v) is a direct consequence of (Foulis, 2003, Theorem 5.1). □

Corollary 3. *A directed abelian group with a finitely generated positive cone is archimedean iff it is unperforated.*

Proof: If a directed abelian group is archimedean, then it is unperforated (Goodearl, 1986, Proposition 1.24). Conversely, suppose that H is a directed unperforated abelian group with a finitely generated positive cone. Without loss of generality, we can assume that $H \neq \{0\}$. Since H is unperforated, it is torsion free. By (Foulis, 2003, Theorem 2.1), H has a generative order unit u and $H^+[0, u]$ is finite. Thus by Theorem 1, there is a unital isomorphism from H onto a group G satisfying our standing assumptions. Since H is unperforated, so is G, whence G is archimedean by Theorem 6, and it follows that H is archimedean. □

As is shown by Example 1, if the hypothesis that the directed abelian group G in Corollary 3 has a finitely generated positive cone is dropped, the conclusion of the corollary may fail even if G is a free group with finite rank and G^+ satisfies the descending chain condition.

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